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INVARIANT IMBEDDING AND SCATTERING OF LIGHT IN A ONE-DIMENSIONAL MEDIUM WITH A MOVING BOUNDARY

Richard Bellman, Robert Kalaba and Sueo Ueno

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The RAND Corporation
SANTA MONICA • CALIFORNIA

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PREFACE

Part of the RAND research program consists of basic supporting studies in mathematics. The present paper continues the work of the authors of invariant-imbedding techniques for solving problems in radiative transport theory.

SUMMARY

In the present paper it is shown how the invariant-imbedding technique is used for the derivation of integral equations governing the reflection and the transmission coefficients of radiation in a one-dimensional medium with a moving boundary. In Part A, when formulating the source function, account is taken by the relaxation of photon emission under the assumption that during an elementary act of scattering a photon remains in a state of absorption for a certain duration of temporal capture. In Part B, the Schuster problem for a moving one-dimensional medium is exactly treated. In other words, when formulating the transfer equations, the influence of the Doppler effect caused by the macroscopic differential motions is considered as a principal agency of the change of frequency on scattering.

CONTENTS

INVARIANT IMBEDDING AND SCATTERING OF LIGHT IN A ONE-DIMENSIONAL MEDIUM WITH A MOVING BOUNDARY

1. INTRODUCTION

In recent years, for analytical and numerical advantages, the principle-of-invariance method originally due to Ambarzumian [1] and Chandrasekhar [6] has been developed by several authors (Bellman, Kalaba, and Wing [2] and [3], Wing [11]). The invariant-imbedding technique thus proposed was successfully applied not only to various kinds of stationary transport problems for photons and neutrons, but also to time-dependent transfer problems. Many previous attempts at solving nonsteady transfer problems will be found in the list of references of the preceding papers (see also Bellman, Kalaba, and Ueno [4] and [5], Kaplan [8], and Wing [10]).

Recently, in the study of astrophysics, emphasis has been put on the elucidation of the photon diffusion process in a differentially moving atmosphere, whereas the exact treatment seems very difficult on account of the untractability of the transfer equation due to the Doppler shift in frequency. Such problems are encountered in the interpretation of spectral lines in novae, planetary nebulae, and the solar prominences and corona. This state of affairs is also found in the analysis of several physical experiments.

By reducing the transfer problem to a novel boundary—value problem for hyperbolic differential equations, Chandrasekhar [7] has given apprcximately the solutions illustrating the effects that may be expected in the rectangular contour of the scattering coefficient. Later, combining the Doppler effect due to the thermal motions of atoms and the macroscopic differential motions in finite one— and three—dimensional media and assuming an arbitrary contour of the scattering coefficient, Sobolev [9] has considered the approximate effect of the complete redistribu'ion of radiation in frequency on the radiation pressure due to the Lyman—α.

In a preceding paper, the invariant—imbedding technique was applied to the study of time—dependent scattering of light in a one—dimensional medium of a finite optical thickness. In this paper, extending the imbedding procedure to a one—dimensional moving medium, we shall consider a couple of nonsteady scattering processes as follows: In Part A the left—hand boundary of the medium moves as a prescribed function of time, and the source function is formulated under consideration of the relaxation of the photon emission. In Part B, we treat the Shuster problem for a differentially moving atmosphere such that, assuming that the only optical effect of large—scale motions is a change of frequency on scattering, the interaction of radiation fields with different frequencies is taken into account. In a later paper, extending this

procedure and allowing for the noncoherent scattering, we shall consider the formation of spectral lines in a differentially moving atmosphere in rod and slab geometries.

A. RELAXATION OF PHOTON EMISSION

2. THE EQUATION OF TRANSFER

Consider a one-dimensional medium of finite geometrical thickness $z_1 - z_0 > 0$ with a moving boundary whose optical properties vary with its position and with time. Let a pencil of radiation of Dirac delta-function time-dependent intensity $I_0(t) = F\,\delta(t - s)$ $(t > s \geq 0)$ be incident on the fixed boundary z_1, where F is a constant and δ is the Dirac delta-function (see Fig. 1). At time t, the position of the left-hand boundary is $z_0 = z_0(t)$, whereas at $t = 0$ it is equal to zero; further, $|dz_0/dt|$ is less than the speed of light in the medium.

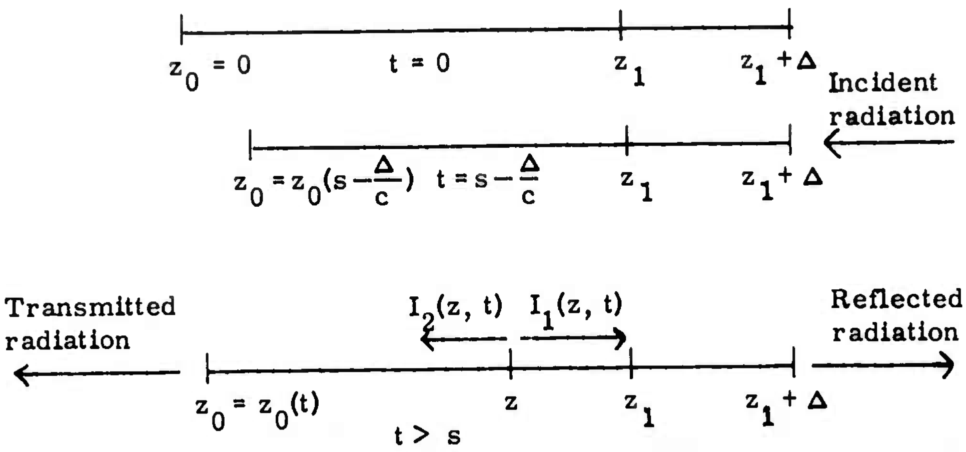

Fig. 1 The reflection and the transmission of radiation in a one-dimensional medium with a moving boundary

For the sake of simplicity, the scattering of light in either direction is assumed to be equally probable.

Let $I_1(z, t)$ denote the specific intensity of radiation at depth z, at time t, directed toward the boundary z_1, and similarly let $I_2(z, t)$ denote that directed toward the boundary $z_0(t)$.

The equation of transfer appropriate to the present case is written in the form

$$\frac{\partial I_1}{\partial z} + \frac{1}{c}\frac{\partial I_1}{\partial z} = -\ell(z, t)\, I_1(z, t) + B(z, t), \tag{2.1}$$

$$-\frac{\partial I_2}{\partial z} + \frac{1}{c}\frac{\partial I_2}{\partial z} = -\ell(z, t)\, I_2(z, t) + B(z, t), \tag{2.2}$$

where c is the speed of light, ℓ is the extinction coefficient, and $B(z, t)$ is the source function.

In a manner similar to that used in a preceding paper [5], the source function takes the form

$$B(z, t) = \frac{1}{2}\,\sigma(z, t)\int_s^t \{I_1(z, t') + I_2(z, t')\}\, e^{-(t-t')/t_1}\,\frac{dt'}{t_1} \tag{2.3}$$

where σ is the scattering coefficient and t_1 is the duration of temporal capture. Equation (2.3) represents the relaxation of emission of absorbed energy.

Equations (2.1) and (2.2) should be solved subject to the boundary and initial conditions

$$I_1(z_0(t), t) = 0, \quad I_2(z_1, t) = F\delta(t-s), \tag{2.4}$$

$$I_1(z, s) = 0, \quad I_2(z, s) = 0 \quad (z_0 \le z < z_1, \; 0 \le s < t), \tag{2.5}$$

where $I_1(z_1, t)$ and $I_2(z_0, t)$ are respectively the reflected and the transmitted intensities.

3. THE REFLECTED INTENSITY

Let the coefficient of reflection be denoted by $R(z_1; t, s)$. Then the reflected intensity is expressed in terms of the reflection coefficient R as follows:

$$I_1(z_1, t) = \int_s^t R(z_1; t, t') I_2(z_1, t')dt', \tag{3.1}$$

where $I_2(z_1, t)$ is given by (2.4).

We shall inquire into an integral equation for $R(z_1; t, s)$ with the aid of the invariant-imbedding technique.

When we add a layer of infinitesimal thickness Δ to the boundary z_1, the imbedding procedure of the scattering in the added layer yields

$$I_1(z_1 + \Delta, t + \frac{\Delta}{c}) = I_1^*(z_1, t) + \{-\ell(z_1, t) I_1(z, t) + B(z_1, t)\}\Delta + \mathcal{O}(\Delta), \tag{3.2}$$

where $\mathcal{O}(\Delta)$ is of the order of magnitude of the infinitesimal Δ^2. Equation (3.2) shows that the reflected intensity at the boundary $z_1 + \Delta$ at time $t + \frac{\Delta}{c}$, due to the incident intensity $I_2(z_1 + \Delta, t')$,

is produced by the intensity at the boundary z_1 at time t, due to a modified incident radiation source $\hat{I}_2(z_1, t)$, in addition to the intensity produced by the interaction of the radiation field in $(z_1, z_1 + \Delta)$.

The term I_1 on the left-hand side of (3.2) is given by

$$(3.3)\quad I_1(z_1 + \Delta,\ t + \frac{\Delta}{c}) = \int_{s - \frac{\Delta}{c}}^{t + \frac{\Delta}{c}} R(z_1 + \Delta; t + \frac{\Delta}{c},\ x)\, I_2(z_1 + \Delta,\ x)dx,$$

where $I_2(z_1 + \Delta,\ t) = F\,\delta(t - s + \frac{\Delta}{c})$.

On the other hand, recalling (2.2), we obtain the modified incident source given by

$$(3.4)\quad \hat{I}_2(z_1,\ t) = I_2(z_1 + \Delta,\ t - \frac{\Delta}{c}) - \ell(z_1,\ t)\, I_2(z_1,\ t)\ell$$
$$+ \frac{1}{2}\,\sigma(z_1,\ t)\,\Delta \int_s^t \{I_1(z_1,\ x) + I_2(z_1,\ x)\}\, e^{-(t - x)/t_1} \frac{dx}{t_1} + \mathcal{O}(\Delta).$$

Thus, the intensity of radiation due to the modified source $\hat{I}_2$ is

$$(3.5)\quad I_1^*(z_1,\ t) = \int_s^t R(z_1;\ t,\ x)\, \hat{I}_2(z_1,\ x)dx$$
$$= \int_s^t R(z_1;\ t,\ x)\, I_2(z_1 + \Delta,\ x - \frac{\Delta}{c})dx - \Delta \int_s^t R(z_1;\ t,\ x)\,\ell(z_1,\ x)\, I_2(z_1,\ x)dx$$
$$+ \frac{1}{2}\,\Delta \int_s^t R(z_1;\ t,\ x)\,\sigma(z_1,\ x)dx \int_s^x I_1(z_1,\ y)\, e^{-(x - y)/t_1} \frac{dy}{t_1}$$

$$+\frac{1}{2}\Delta\int_s^t R(z_1;\, t,\, x)\,\sigma(z_1,\, x)dx\int_s^x I_2(z_1,\, y)\, e^{-(x-y)/t_1}\frac{dy}{t_2} + \mathcal{O}(\Delta).$$

On substituting (3. 3) and (3. 5) into (3. 2) and putting $I_2(z_1,\, t) = I_0(t)$, we get

$$(3.6)\quad R(z_1+\Delta;\, t+\frac{\Delta}{c},\, s-\frac{\Delta}{c}) = R(z_1;\, t,\, s) - \ell(z_1,\, s)\, R(z;\, t,\, s)\Delta$$
$$+\frac{1}{2}\Delta\int_s^t R(z_1;\, t,\, x)\,\sigma(z_1,\, x)dx\int_s^x R(z_1;\, y,\, x)\, e^{-(x-y)/t_1}\frac{dy}{t_1}$$
$$+\frac{1}{2}\Delta\int_s^t R(z_1;\, t,\, x)\,\sigma(z_1,\, x)\, e^{-(x-s)/t_1}\frac{dx}{t_1}$$
$$+\Delta\Big[-\ell(z_1,\, t)\, R(z_1;\, t,\, s) + \frac{1}{2}\sigma(z_1,\, t)\int_s^t R(z_1;\, x,\, s)\, e^{-(t-x)/t_1}\frac{dx}{t_1}$$
$$+\frac{1}{2}\sigma(z_1,\, t)\,\frac{e^{-(t-s)/t_1}}{t}\Big] + \mathcal{O}(\Delta).$$

Then, in the limit as $\Delta \to 0$, (3. 6) becomes

$$(3.7)\quad \frac{\partial R}{\partial z_1} + \frac{1}{c}\frac{\partial R}{\partial t} - \frac{1}{c}\frac{\partial R}{\partial s} + (\ell(z_1,\, t) + \ell(z_1,\, s))R$$
$$= \frac{1}{2}\,\sigma(z_1,\, t)\,\frac{e^{-(t-s)/t_1}}{t_1} + \frac{1}{2}\sigma(z_1,\, t)\int_s^t R(z_1;\, x,\, s)\, e^{-(t-x)/t_1}\frac{dx}{t_1}$$
$$+\frac{1}{2}\int_s^t R(z_1;\, t,\, x)\,\sigma(z_1,\, x)\, e^{-(x-x)/t_1}\frac{dx}{t_1}$$
$$+\frac{1}{2}\int_s^t R(z_1;\, t,\, x)\,\sigma(z_1,\, x)dx\int_s^x R(z_1;\, y,\, s)\, e^{-(x-y)/t_1}\frac{dy}{t_1}.$$

Equation (3. 7) is the requisite integral equation for the reflection coefficient. The boundary and initial conditions imposed on R are given by

$$(3.8)\qquad R(z_1;\, t,\, s) = 0 \quad \text{for} \quad z_0(s) = z_1,$$

$$(3.9)\qquad R(z_1;\, t,\, s) = 0 \quad \text{for} \quad s \geq t \geq 0.$$

When the optical properties of the medium are steady and homogeneous with respect to time and depth, and we put

$$(3.10)\qquad \tau = \ell z, \quad t_2 = \frac{1}{c\ell}, \quad \tilde{\omega} = \frac{\sigma}{\ell},$$

where τ is the optical depth, t_2 is the mean free time, and $\tilde{\omega}$ is the albedo for single scattering, (3.7) reduces to

$$(3.11)\qquad \frac{\partial R}{\partial \tau_1}(\tau_1;\, t-s) + t_2\frac{\partial R}{\partial t} - t_2\frac{\partial R}{\partial s} + 2R$$
$$= \frac{\tilde{\omega}}{2}\left[\frac{e^{-(t-s)/t_1}}{t_1} + 2\int_s^t R(\tau_1;\, t-x)\, e^{-(x-s)/t_1}\frac{dx}{t_1}\right.$$
$$\left. + \int_s^t R(\tau_1;\, t-x)dx\int_s^x R(\tau_1;\, y-s)\, e^{-(x-y)/t_1}\frac{dy}{t_1}\right].$$

4. THE TRANSMITTED INTENSITY

Consider the case of illumination of the fixed boundary z_1. Let the coefficient of transmission be denoted by $T(z_1;\, z_0(t),\, t;\, z_0(s),\, s)$. Then the transmitted intensity at the boundary z_0 at time t is expressed in the form

$$(4.1)\qquad I_2(z_0(t),\, t) = \int_s^{t^*} T(z_1;\, z_0(t),\, t;\, z_0(x),\, x)\, I_2(z_1,\, x)dx,$$

where $I_2(z_1,\, t)$ is given by (2.4) and $t^* = t - (z_1 - z_0(t))/c$.

For convenience, we shall write

$$(4.2)\qquad T(z_1;\, z_0(t),\, t;\, z_0(s),\, s) = T(z_1;\, t,\, s).$$

In what follows, we shall seek an integral equation for the T-function. On adding a layer of infinitesimal thickness Δ to the boundary z_1, we get (3.4). On applying the imbedding argument to the transmitted intensity, we obtain

$$(4.3)\qquad I_2^0(z_0(t),\, t) = \overset{*}{I}_2(z_0(t),\, t).$$

On the other hand, we have

$$(4.4)\qquad I_2^0(z_0(t),\, t) = \int_{s-\frac{\Delta}{c}}^{t^*-\Delta/c} T(z_1+\Delta;\, t,\, x)\, I_2(z_1+\Delta,\, x)dx,$$

and

$$(4.5)\qquad \overset{*}{I}_2(z_0(t),\, t) = \int_s^{t^*} T(z_1;\, t,\, x)\, \hat{I}_2(z_1,\, x)dx.$$

Substitution of (3.4) into (4.5) yields

$$\begin{aligned}(4.6)\quad \overset{*}{I}_2(z_0(t),\, t)/F &= T(z_1;\, t,\, s) - \ell(z_1,\, s)\, T(z_1;\, t,\, s)\Delta \\ &+ \frac{1}{2}\Delta\int_s^{t^*} T(z_1;\, t,\, x)\sigma(z_1,\, x)\, e^{-(x-s)/t_1}\frac{dx}{t_1} \\ &+ \frac{1}{2}\Delta\int_s^{t^*} T(z_1;\, t,\, x)\,\sigma(z_1,\, x)dx\int_s^x R(z_1;\, y,\, s)\, e^{-(x-y)/t_1}\frac{dy}{t_1} + O(\Delta).\end{aligned}$$

On combining (2.4) and (4.3)-(4.5), we get

$$(4.7)\quad T(z_1+\Delta;\, t,\, s-\frac{\Delta}{c}) = T(z_1;\, t,\, s) - \ell(z_1,\, s)\, T(z_1;\, t,\, s)\Delta$$

$$+\frac{1}{2}\Delta\int_s^{t*} T(z_1;\, t,\, s)\,\sigma(z_1,\, x)\, e^{-(x-s)/t_1}\frac{dx}{t_1}$$

$$+\frac{1}{2}\Delta\int_s^{t*} T(z_1;\, t,\, x)\,\sigma(z_1,\, x)dx\int_s^x R(z_1;\, y,\, s)\, e^{-(x-y)/t_1}\frac{dy}{t_1}$$

$$+\mathcal{O}(\Delta).$$

Letting $\Delta \to 0$ in (4.6) gives

$$(4.8)\quad \frac{\partial T}{\partial z_1} - \frac{1}{c}\frac{\partial T}{\partial s} + \ell(z_1,\, s)T$$

$$= \frac{1}{2}\int_s^{t*} T(z_1;\, t,\, x)\,\sigma(z_1,\, x)\, e^{-(x-s)/t_1}\frac{dx}{t_1}$$

$$+\frac{1}{2}\int_s^{t*} T(z_1;\, t,\, x)\,\sigma(z_1,\, x)dx\int_s^x R(z_1;\, y,\, s)\, e^{-(x-y)/t_1}\frac{dy}{t_1}\,.$$

Equation (4.8) is the required integral equation governing the transmission coefficient, where $T(z_1;\, t,\, s) = 0$ for

$$t - s \le \frac{z_1 - z_0(t)}{c}\,.$$

Let the reduced and the diffuse components of the transmission coefficient be denoted by $T^o(z_1;\, t,\, s)$ and $T^*(z_1;\, t,\, s)$, respectively, where

$$(4.9)\quad T^o(z_1;\, t,\, s) = \exp\left[-\int_{z_0(t)}^{z_1} \ell(z,\, s+\frac{z_1-z}{c}\, dz\right]\delta(t - \frac{z_1 - z_0(t)}{c} - s).$$

Then the insertion of $T = T^0 + T^*$ into (4.8) provides us with an integral equation for the T^*–function.

In the steady and homogeneous case with respect to time and depth, by (3.10), (4.8) becomes

$$(4.10)\quad \frac{\partial T}{\partial \tau_1}(\tau_1;\, t-s) - t_2\frac{\partial T}{\partial s} + T$$
$$= \frac{\tilde{\omega}}{2}\left[\int_s^{t^*} T(\tau_1;\, t-x)\, e^{-(x-s)/t_1}\frac{dx}{t_1}\right.$$
$$\left. + \int_s^{t^*} T(\tau_1;\, t-x)dx \int_s^x R(\tau_1;\, y-s)\, e^{-(x-y)/t_1}\frac{dy}{t_1}\right],$$

where $t^* = t - (\tau_1 - \tau_0)t_2$.

B. CHANGE OF FREQUENCY ON SCATTERING DUE TO DOPPLER EFFECT

5. THE EQUATION OF TRANSFER

Consider a one–dimensional, differentially expanding medium of finite geometrical thickness $z_1 - z_0 > 0$ whose velocity $v(z) > 0$ at z is a linear function of the depth, $z_0 \leq z \leq z_1$. Let $\sigma(\nu, z)$ denote the monochromatic volume scattering coefficient at z for the frequency ν as judged by a stationary observer. The observer may be considered to be at rest with respect to the position at $z = 0$. For simplicity, the quantity σ is assumed to be $\rho(z)\,\sigma^*(\nu)$, where $\rho(z)$ is the mass density and $\sigma^*(\nu)$ is the mass scattering coefficient. Furthermore, we assume the characteristic feature of σ^*, that the only optical effect of the large–scale motions $v(z)$ is a change of frequency on scattering.

Let $I_1(\nu, z)$ denote the specific intensity of radiation at z directed toward the boundary z_1 and of frequency ν as judged by the stationary observer, and further let $I_2(\nu, z)$ denote the specific intensity directed toward the boundary z_0 (see Fig. 2).

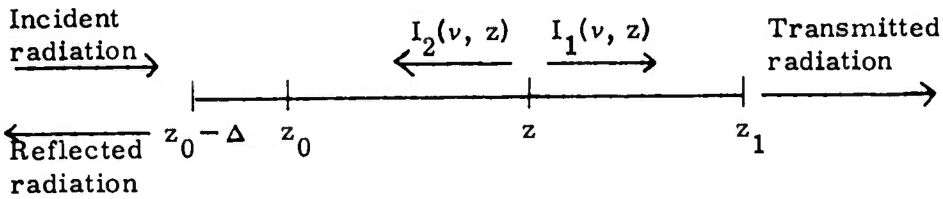

Fig. 2. The reflected and transmitted intensities of radiation by a differentially moving one–dimensional medium

Consider a pencil of radiation of constant intensity F in all frequencies to be incident on z_0. When we take into account the Doppler shift in frequency, $-\frac{\nu}{c}\, v(z)\,(1 \pm 1)$, caused by the large–scale motions, the equations of transfer appropriate to the present case take the form (cf. Chandrasekhar [7])

$$(5.1) \qquad \frac{\partial I_1(\nu, z)}{\partial z} = -\frac{1}{2}\sigma\left(\nu\left(1 - \frac{V}{c}\right), z\right)\left[I_1(\nu, z) - I_2\left(\nu\left(1 - 2\frac{V}{c}\right), z\right)\right],$$

$$(5.2) \qquad -\frac{\partial I_2(\nu, z)}{\partial z} = -\frac{1}{2}\sigma\left(\nu\left(1 + \frac{V}{c}\right), z\right)\left[I_2(\nu, z) - I_1\left(\nu\left(1 + 2\frac{V}{c}\right), z\right)\right],$$

where c is the speed of light.

In what follows, in a manner similar to that used by Chandrasekhar [7], we shall make some simplifications of the transfer equations. In the evaluation of the Doppler shift in frequency, $\nu(1 - \frac{V}{c})$ is reasonably approximated by $\nu - \nu_0 \frac{V}{c}$, where ν_0 is the

frequency of the line center.

Furthermore, in place of considering the equations of transfer for some fixed frequency ν, we shall somewhat modify them in such a way that ν in (5.1) and ν in (5.2) are respectively replaced by ν_1 and ν_2, where

$$(5.3) \qquad \nu_1 = \nu + \nu_0 \frac{v}{c}, \qquad \nu_2 = \nu - \nu_0 \frac{v}{c}.$$

Allowing for (5.3), (5.1) and (5.2) are rewritten as follows:

$$(5.4) \qquad \frac{\partial I_2}{\partial z}(\nu_1, z) = -\frac{1}{2}\sigma(\nu, z)[I_1(\nu_1, z) - I_2(\nu_2, z)],$$

$$(5.5) \qquad -\frac{\partial I_2}{\partial z}(\nu_2, z) = -\frac{1}{2}\sigma(\nu, z)[I_2(\nu_2, z) - I_1(\nu_1, z)].$$

Writing

$$(5.6) \qquad I_i(\nu_i, z) = \Psi_i(\nu, z) \qquad (i = 1, 2),$$

we have

$$(5.7) \qquad \frac{\partial I_i}{\partial z}(\nu_i, z) = \frac{\partial \Psi_i}{\partial z} \mp Q \frac{\partial \Psi_i}{\partial \nu} \qquad \text{for } i = \begin{cases} 1 \\ 2 \end{cases},$$

where Q is a constant given by

$$(5.8) \qquad Q = \frac{\nu_0}{c} \frac{dv(z)}{dz}.$$

By (5.7), (5.4) and (5.5) become

$$(5.9) \qquad \frac{\partial \Psi_1}{\partial z} - Q \frac{\partial \Psi_1}{\partial \nu} = -\frac{1}{2}\sigma(\nu, z)[\Psi_1 - \Psi_2],$$

$$(5.10) \qquad \frac{\partial \Psi_2}{\partial z} + Q \frac{\partial \Psi_2}{\partial \nu} = -\frac{1}{2}\sigma(\nu, z)[\Psi_1 - \Psi_2].$$

Equations (5.9) and (5.10) should be solved subject to the boundary conditions

$$(5.11) \qquad \Psi_1(\nu, z_0) = F, \qquad \Psi_2(\nu, z_1) = 0,$$

where F is a constant for all frequencies.

The quantities $I_2(\nu, z_0)$ and $I_1(\nu, z_1)$ represent respectively the reflected and the transmitted intensities.

6. THE REFLECTION COEFFICIENT

Let the coefficient of reflection be denoted by $R(z_0, z_1; \nu)$. Then the reflected intensity $\Psi_2(\nu, z_0)$ is expressed in terms of R as follows:

$$(6.1) \qquad \Psi_2(\nu, z_0) = F\, R(z_0, z_1; \nu).$$

In a manner similar to that used in Sec. 3, we shall seek an integral equation for the R–function by means of the invariant–imbedding technique.

By addition of a layer of infinitesimal thickness Δ to the boundary z_0, and use of (5.10), the imbedding procedure gives

$$(6.2)\quad \Psi_2(\nu - Q\Delta, z_0 - \Delta) = \Psi_2^*(\nu, z_0) + \frac{1}{2}\sigma(\nu, z_0)\Delta(\Psi_1(\nu, z_0) - \Psi_2(\nu, z_0)) + O(\Delta).$$

By (5.9), the modified intensity at z in an added layer is provided by

$$(6.3)\quad \hat{\Psi}_1(\nu, z_0) = \Psi_1(\nu + Q\Delta, z_0 - \Delta) - \frac{1}{2}\sigma(\nu, z_0)\Delta[\Psi_1(\nu, z_0) - \Psi_2(\nu, z_0)] + O(\Delta).$$

On making use of (6.3), we see that the reflected intensity of radiation due to the modified source $\hat{\Psi}_1$ becomes

$$(6.4)\quad \Psi_2^*(\nu, z_0) = F\,[R(z_0, z_1; \nu) - \frac{1}{2}\sigma(\nu, z_0)R(z_0, z_1; \nu)\Delta$$
$$+ \frac{1}{2}\sigma(\nu, z_0)R^2(z_0, z_1; \nu)\Delta] + O(\Delta).$$

The combination of (6.1), (6.2), and (6.4) provides us with

$$(6.5)\quad R(z_0 - \Delta, z_1; \nu - Q\Delta) = R(z_0, z_1; \nu)$$
$$+ \frac{1}{2}\sigma(\nu, z_0)\Delta - \sigma(\nu, z_0)R(z_0, z_1; \nu)\Delta + \frac{1}{2}\sigma(\nu, z_0)R^2(z_0, z_1; \nu)\Delta + O(\Delta).$$

Then, letting $\Delta \to 0$, we have

$$(6.6)\quad \frac{\partial R}{\partial z_0} + Q\frac{\partial R}{\partial \nu} = -\frac{1}{2}\sigma(\nu, z_0) + \sigma(\nu, z_0)R$$
$$- \frac{1}{2}\sigma(\nu, z_0)R^2.$$

Equation (6.6) is the required integral equation for the reflection coefficient R.

7. THE TRANSMISSION COEFFICIENT

Consider the transmitted intensity of radiation emergent from the boundary z_1 for the illumination of the boundary z_0.

Let the coeffi[illegible]ient of transmission be denoted by $T(z_0, z_1; \nu)$. The transmitted intensity at z_1 is expressed in the form

$$(7.1) \qquad \Psi_1(\nu, z_1) = F\, T(z_0, z_1; \nu).$$

On adding a layer of infinitesimal thickness Δ to the boundary z_0, we get (6.3). By imbedding the photon diffusion process in the added Δ-layer, we obtain

$$(7.2) \qquad \Psi_1^{0}(\nu, z_1) = \Psi_1^{*}(\nu, z_1).$$

On the other hand, we have

$$(7.3) \qquad \Psi_1^{0}(\nu, z_1) = F\, T(z_0 - \Delta, z_1; \nu + Q\Delta),$$

and

$$(7.4) \qquad \Psi_1^{*}(\nu, z_1) = T(z_0, z_1; \nu)\, \hat{\Psi}_1(\nu, z_0).$$

Then, combining (6.3) and (7.3)-(7.4), we obtain (7.2) in the form

$$(7.5) \qquad T(z_0 - \Delta, z_1; \nu + Q\Delta) = T(z_0, z_1; \nu)\, [1 - \\ - \frac{1}{2}\sigma(\nu, z_0)\Delta\{1 - R(z_0, z_1; \nu)\}] + \mathcal{O}(\Delta).$$

In the limit as $\Delta \rightarrow 0$, we have

$$\frac{\partial T}{\partial z_0} - Q \frac{\partial T}{\partial \nu} = \frac{1}{2}\,\sigma(\nu, z_0)\, T - \frac{1}{2}\,\sigma(\nu, z_0)\, T\, R. \tag{7.6}$$

Equation (7.6) is the requisite integral equation governing the transmission coefficient T.

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